Hunters Hill High School

Mathematics Extension 1

Trial Examination, 2016

Hunters Hill High School

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided
- The marks for each question are shown on the paper
- Show all necessary working in questions 11-14.

Total Marks: 70

Section I Pages 3-6
10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II Pages 7-12
60 marks

- Attempt Questions 11-14
- Allow about 1 hour 45 minutes for this section

Section I

10 marks Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1. The coefficient of x^4 in $(2x-3)^{13}$ is

(A) $-\binom{13}{4}2^4 3^9$

(B) $-\binom{13}{4}2^9 3^4$

(C) $\binom{13}{9}2^4 3^9$

(D) $\binom{13}{9}2^9 3^4$

2. What is the domain and range of $y = \sin^{-1}\left(\frac{2x}{5}\right)$?

(A) Domain: $-1 \leq x \leq 1$; Range: $-\pi \leq y \leq \pi$

(B) Domain: $-1 \leq x \leq 1$; Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(C) Domain: $-\frac{5}{2} \leq x \leq \frac{5}{2}$; Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(D) Domain: $-\frac{5}{2} \leq x \leq \frac{5}{2}$; Range: $-\pi \leq y \leq \pi$

3. A particle moves in a straight line. Its position at any time, t, is given by

$$x = 3\cos 2t + 4\sin 2t.$$

The acceleration in terms of x is:

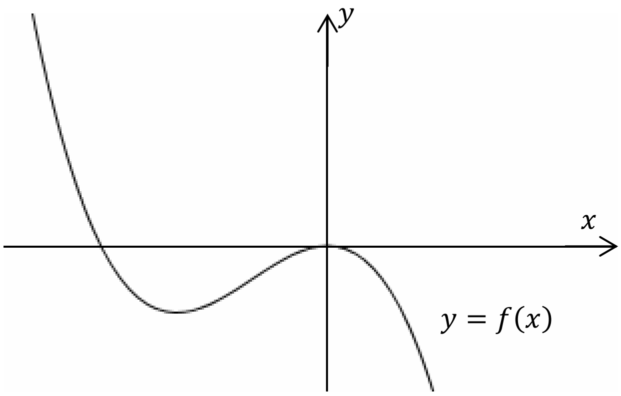
(A) $\ddot{x} = -3x$

(B) $\ddot{x} = -4x$

(C) $\ddot{x} = -16x^2$

(D) $\ddot{x} = -6\cos 2x + 8\sin 2x$

4. The diagram below shows the graph of a cubic function $y = f(x)$.



Which is a possible equation of this function?

(A) $f(x) = -x(x-2)(x+2)$

(B) $f(x) = x^2(x-2)$

(C) $f(x) = -x^2(x+2)$

(D) $f(x) = -x^2(x-2)$

5. What is the equation of the tangent at the point $(4p, 2p^2)$ on the parabola $x^2 = 8y$?

(A) $y = px - p^2$

(B) $x + py = 2p + p^3$

(C) $x + py = 4p + p^3$

(D) $y = px - 2p^2$

6. The equation(s) of the horizontal asymptote(s) to the curve $y = \dfrac{x^2+1}{x^2-1}$ is/are:

(A) $y = 0$

(B) $x = \pm 1$

(C) $y = 1$

(D) $x = 1$ only

7. Which of the following statements is FALSE?

(A) $\cos^{-1}(-\theta) = -\cos^{-1}\theta$

(B) $\sin^{-1}(-\theta) = -\sin^{-1}\theta$

(C) $\tan^{-1}(-\theta) = -\tan^{-1}\theta$

(D) $\cos^{-1}(-\theta) = \pi - \cos^{-1}\theta$

8. Three Mathematics study guides, four Mathematics textbooks and five exercise books are randomly placed along a bookshelf. What is the probability that the Mathematics textbooks are all next to each other?

(A) $\dfrac{4!}{12!}$

(B) $\dfrac{9!}{12!}$

(C) $\dfrac{4!\,3!\,5!}{12!}$

(D) $\dfrac{4!\,9!}{12!}$

9. Water pours into a cylindrical tank of radius 50 cm at a rate of 1.5 L/s.
The rate at which the height of the water in the tank is changing, with respect to time, is:

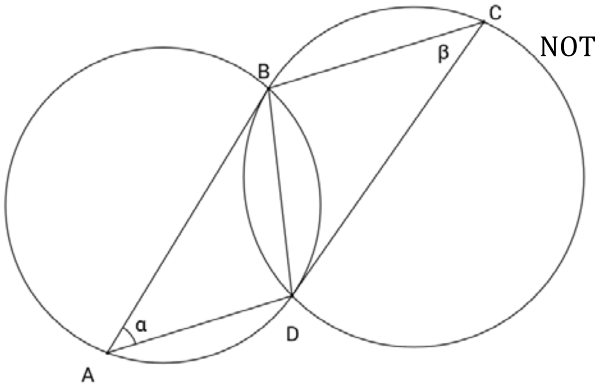
(A) $\dfrac{3}{5000\pi}$ cm s^{-1}

(B) 15π cm s^{-1}

(C) $\dfrac{3}{\pi}$ cm s^{-1}

(D) $\dfrac{3}{5\pi}$ cm s^{-1}

10. In the diagram below, AB is a tangent to the circle BCD, and CD is a tangent to the circle ABD. $\angle BAD = \alpha$ and $\angle BCD = \beta$.



Which of the following statements is true?

(A) $\Delta ABD \equiv \Delta BDC$

(B) $ABCD$ is a cyclic quadrilateral

(C) $\Delta ABD \,|||\, \Delta BDC$

(D) $AB \parallel CD$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section
Begin each question on a NEW SHEET of paper.

In questions 11 – 14 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a. Let $A = (-1, 4)$ and $B = (5, -5)$. Find the coordinates of the point P which divides the interval AB in a ratio of $1:2$. **2**

b. A family consists of 8 people, including three young children.

i. In how many ways can the family be arranged around a table so that the children sit together? **2**

ii. What is the probability that if randomly allocated a seat at the table, the children all sit together? **2**

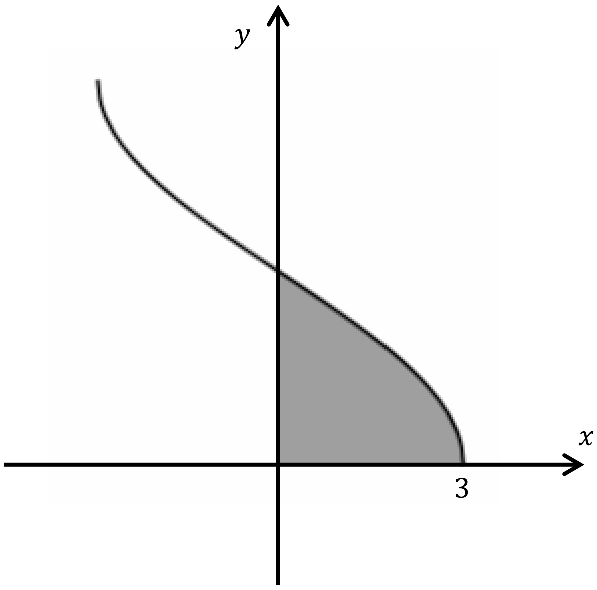
c. Solve the inequality **2**

$$\frac{1}{|x-1|} > \frac{1}{2}$$

d. Differentiate $e^{\cos x} \ln x$ **2**

Question 11 continued on next page

e. The sketch shows the graph of the curve $y = 2\cos^{-1}\frac{x}{3}$.



i. Find the y-intercept. **1**

ii. State the domain and range of the function. **2**

iii. Calculate the area of the function across the domain $0 \leq x \leq 3$ (the shaded region). **2**

End of Question 11

Question 12 (15 marks) Begin a SEPARATE sheet of paper.

a. Let α, β, and γ be the roots of the equation $x^3 - 2x^2 + 5x - 1 = 0$.
Find:

i. $2\alpha + 2\beta + 2\gamma$ **1**

ii. $\alpha^2 + \beta^2 + \gamma^2$ **2**

b. How many 4-letter "words' consisting of at least one vowel and at least one consonant can be made from the letters of the word EQUATION? **2**

c. The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.
The equation of the chord PQ is $y = \frac{1}{2}(p+q)x - apq$.
(Do NOT prove this)

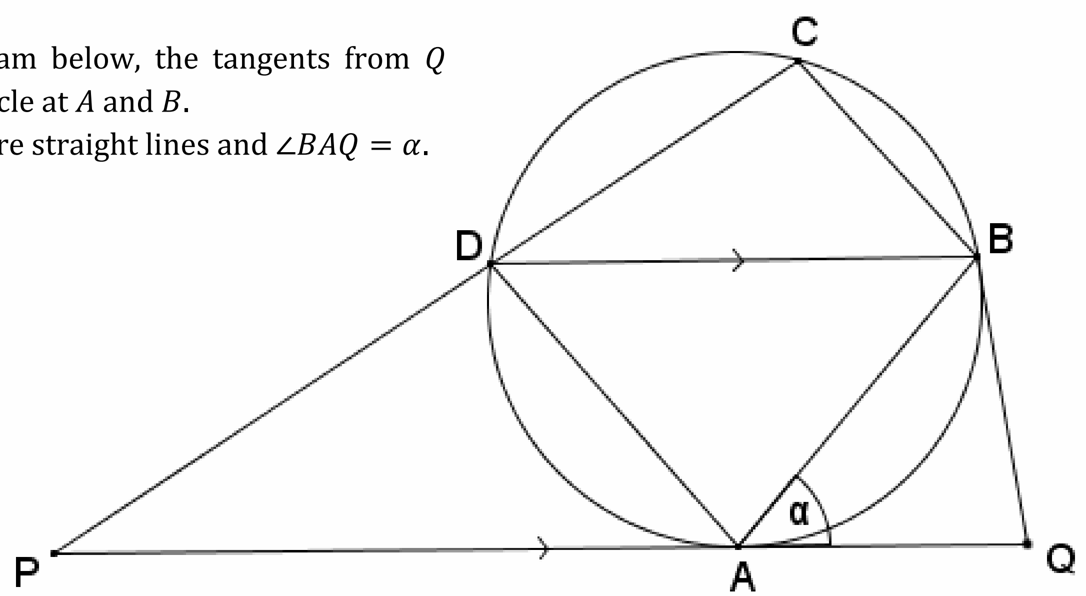
i. If the chord PQ passes through the focus, show that $pq = -1$. **1**

ii. Further, the normals of P and Q intersect at the point R whose coordinates are

$$\left(-apq(p+q), a(p^2 + pq + q^2 + 2)\right)$$

Find the equation of the locus of R. **2**

d. In the diagram below, the tangents from Q touch the circle at A and B.
PC and PQ are straight lines and $\angle BAQ = \alpha$.



i. Copy or trace the diagram onto your writing paper. **1**

ii. Given $PD = 5$cm and $DC = 7$cm, calculate the exact length of AP. **1**

iii. Show that $\angle BCD = 2\alpha$. **3**

iv. Show that $PQBC$ is a cyclic quadrilateral. **2**

End of Question 12

Question 13 (15 marks) Begin a SEPARATE sheet of paper.

a. The velocity of a particle is given as $v^2 = 24 - 6x - 3x^2$.

i. Find the particle's greatest displacement. **1**

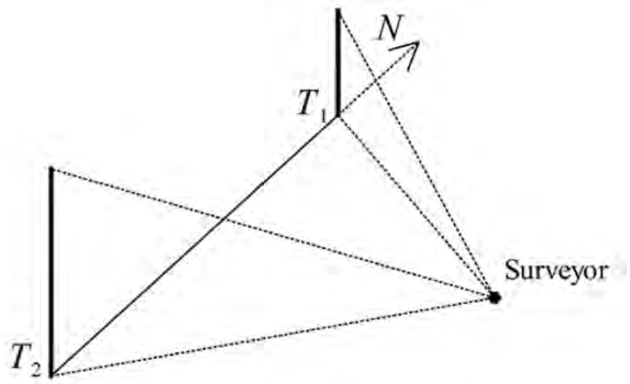
ii. Hence, find the acceleration of the particle at its greatest displacement from origin. **2**

b. Find by division of polynomials, the remainder when $x^2 + 4$ is divided by $x - 3$. **2**

c. Use mathematical induction to prove that $3^n + 7^{n+1}$ is divisible by 4 for all integers $n \geq 1$. **3**

d. Evaluate $\int_0^{\frac{\pi}{4}} \sin^2 2x \, dx$ **3**

e. Two towers T_1 and T_2 have heights of h and $2h$ respectively. Tower T_2 is due south of T_1. The bearing of tower T_1 from a surveyor is 292°, whilst the bearing of T_2 from the surveyor is 232°.
The angle of elevation from the surveyor to the top of tower T_1 is 30° and the angle of elevation from the surveyor to the top of tower T_2 is 60°.



Show that the distance d between the two towers is given by **4**

$$d = \frac{\sqrt{21}h}{3} \text{ metres.}$$

End of Question 13

Question 14 (15 marks) Begin a SEPARATE sheet of paper.

a. $f(x) = \sin^2 x - x + 1$ has a zero near $x_1 = \frac{\pi}{2}$. **2**

Use one application of Newton's method to obtain another approximation, x_2, to this zero.

b. Use the substitution $u = \tan x$ to find an expression for **3**

$$\int \frac{\sec^2 x}{\tan^2 x + 3} dx$$

c. The binomial theorem is

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = (1 + x)^n$$

where $c_k = \binom{n}{k}$

i. Show that **2**

$$\frac{c_0 x^2}{1.2} + \frac{c_1 x^3}{2.3} + \frac{c_2 x^4}{3.4} + \cdots + \frac{c_n x^{n+2}}{(n+1)(n+2)} = \frac{(1+x)^{n+2}}{(n+1)(n+2)}$$

ii. Hence, find the following sum, writing your answer as a simple expression in terms of n. **1**

$$\frac{c_0}{1.2} - \frac{c_1}{2.3} + \frac{c_2}{3.4} + \cdots + (-1)^n \frac{c_n}{(n+1)(n+2)}$$

Question 14 continues on next page.

d. A fire hose is at ground level on a horizontal plane. Water is projected from the hose. The angle of projection, θ, is allowed to vary. The speed of the water as it leaves the hose, v metres per second, remains constant.

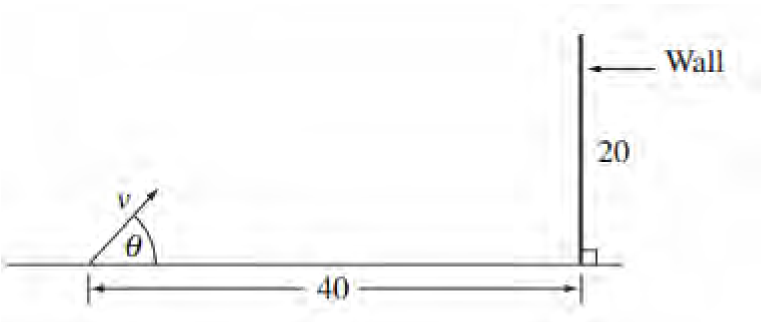
You may assume that if the origin is taken to be the point of projection, the path of the water is given by the parametric equations

$$x = vt\cos\theta$$
$$y = vt\sin\theta - \frac{1}{2}gt^2$$

where g ms^{-2} is the acceleration due to gravity. (Do NOT prove this)

i. Show that the water returns to ground level at a distance $\dfrac{v^2 \sin 2\theta}{g}$ metres from the point of projection. **2**

This fire hose is now aimed at a 20 metre high thin wall from a point of projection at ground level 40 metres from the base of the wall. It is known that when the angle θ is 15°, the water just reaches the base of the wall.



ii. Show that $v^2 = 80g$. **1**

iii. Show that the Cartesian equation of the path of the water is given by **2**

$$y = x\tan\theta - \frac{x^2\sec^2\theta}{160}.$$

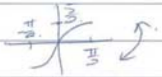
iv. Show that the water just clears the top of the wall if **2**

$$\tan^2\theta - 4\tan\theta + 3 = 0.$$

End of paper

2016 TRIAL - MAX - SOLUTIONS

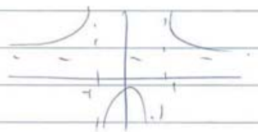
1. $\binom{13}{4} 2^4 (-3)^9 = -\binom{13}{4} 2^4 3^9$ (A)

2. $y = \frac{5}{2}\sin x$  (C)

3. $x = 3\cos 2t + 4\sin 2t$
$\dot{x} = -6\sin 2t + 8\cos 2t$
$\ddot{x} = -12\cos 2t - 16\sin 2t$
$= -4(x)$ (B)

4. (C)

5. $\frac{dy}{dx} = \frac{x}{4} = \frac{4p}{4} = p$ (D)
$y - 2p^2 = p(x - 4p)$
$y = px - 2p^2$

6.  (C)

7. (A)

8. total $= 12!$ (D)
ways to group 4 together $= 4!9!$

9. $\frac{dV}{dt} = 1.5$ $\quad V = \pi r^2 h$
$= 2500\pi h$
$h = \frac{V}{2500\pi}$
$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$ (A)
$= \frac{1}{2500\pi} \cdot \frac{3}{2}$
$= \frac{3}{5000\pi}$

10.  (C)
equiangular

Q11. a) $A=(-1,4)$, $B=(5,-5)$ 2.

$1:2$

$x=\frac{1(5)+2(-1)}{1+2}$ $y=\frac{1(-5)+2(4)}{1+2}$

$=\frac{3}{3}$ $=\frac{3}{3}$

$=1$ $=1$

$\therefore P=(1,1)$

b. i.  $5!.3!=120\times6$ 2.

$=720$

ii. $\frac{5!3!}{7!}=\frac{6}{6.7}=\frac{1}{7}$ 2.

c. $\frac{1}{|x-1|}>\frac{1}{2}$ 2.

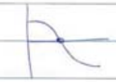
$|x-1|<2$

$-2<x-1<2$

$\therefore -1<x<3$, $x\neq1$

d. $\frac{d}{dx}e^{\cos x}\ln x = e^{\cos x}\cdot\frac{1}{x}+(-\sin x)e^{\cos x}\cdot\ln x$ 2.

e. i y-int at $x=0$ 1.

$\cos\frac{y}{2}=0$ 

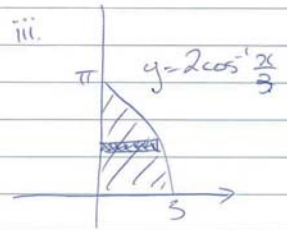
$\frac{y}{2}=\frac{\pi}{2}$

$y=\pi$

ii. domain: $-3\leq x\leq3$ 2.

range: $0\leq y\leq2\pi$

iii. 2.



$V=\int_0^{\pi}x\,dy$

$=3\int_0^{\pi}\cos\frac{y}{2}\,dy$

$=3\left[2\sin\frac{y}{2}\right]_0^{\pi}$

$=6\left(\sin\frac{\pi}{2}-\sin\frac{0}{2}\right)$

$=6(1-0)$

$=6$ units^2

Q12

a. $x^3 - 2x^2 + 5x - 1 = 0$

i. $2\alpha + 2\beta + 2\gamma = 2(\alpha + \beta + \gamma)$ 1.

$= 2\left(- \frac{-2}{1}\right)$

$= 4$.

ii. $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ 2.

$= 2^2 - 2(5)$

$= -6$

b. EQUATION 5V, 3C. 2

~~V. C. _ _~~

$^8P_4 - {}^5P_4$

~~5 . 3 . 6 . 5 × 4! = 10800~~

$= 1560$.

c. PQ: $y = \frac{1}{2}(p+q)x - apq$.

i) chord passes through (0, a) 1.

$\therefore a = \frac{1}{2}(p+q)0 - apq$

$a = -apq$

$pq = -1$

ii 2.

$x = -apq(p+q)$

$= -a(-1)(p+q)$

$p + q = \frac{x}{a}$.

$y = a(p^2 + pq + q^2 + 2)$

$= a(p^2 + 2pq + q^2 + 2 - pq)$

$= a((p+q)^2 + 2 - pq)$

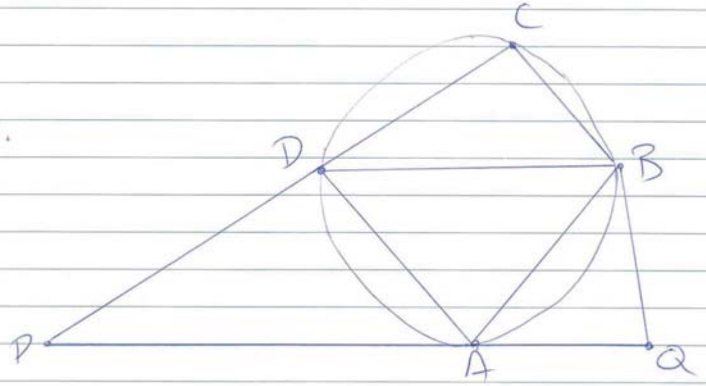
$= a\left(\left(\frac{x}{a}\right)^2 + 2 - (-1)\right)$

$= a\left(\frac{x^2}{a^2} + 3\right)$

$x^2 = a(y - 3a)$

d.

i. 1.



ii. $AP^2 = PD.PC$ (intersecting tangent and secant) 1.

$= 5.(5+7)$

$= 60$

$AP = 2\sqrt{15}$ units

iii. $\angle ADB = \angle BAQ$ (angle in alternate segment)
$= \alpha$.
$\angle DBA = \angle BAQ$ (alternate angles in parallel lines)
$= \alpha$

$\angle DAB + \angle ADB + \angle DBA = 180°$ (angle sum of triangle)
$\angle DAB + \alpha + \alpha = 180°$
$\angle DAB = 180 - 2\alpha$.

$\angle BCD + \angle DAB = 180$ (opposite angles of cyclic quadrilateral)
$\therefore \angle BCD = 180 - (180 - 2\alpha)$
$= 2\alpha$.

iv. $BQ = AQ$. (tangents to a point)
$\angle ABQ = BAQ$ (equal angles of isosceles triangle)
$= \alpha$
$\angle AQB + \alpha + \alpha = 180°$ (angle sum of triangle)
$\therefore \angle AQB = 180 - 2\alpha$

$\angle AQB + \angle BCD = 180 - 2\alpha + 2\alpha$
$= 180°$

$\therefore$ PQBC is a cyclic quadrilateral (opposite angles equal)

Q13. a. $v^2 = 24 - 6x - 3x^2$
i. greatest displacement when $v=0$ 1.

$24 - 6x - 3x^2 = 0$
$x^2 + 2x + 1 = 8 + 1$
$(x+1)^2 = 9$
$x + 1 = \pm 3$
$x = -1 \pm 3$
$= -4, 2$.

greatest displacement is 4 units

ii. $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ 2.
$= \frac{d}{dx}\left(\frac{1}{2}(24 - 6x - 3x^2)\right)$
$= \frac{1}{2}(-6 - 6x)$
$= -3 - 3x$.
at $x = -4$,
$a = -3 - 3(-4)$
$= 9$ units/s^2

b. 2.

$$\begin{array}{r} x + 3 \\ x-3\overline{)\,x^2 \qquad + 4} \\ \underline{x^2 - 3x \qquad} \\ 3x + 4 \\ \underline{3x - 9} \\ 13 \end{array}$$

remainder is 13.

c. Prove $3^n + 7^{n+1}$ is divisible by 4, for $n \geq 1$.

prove true for $n = 1$

$$3^1 + 7^{1+1} = 3 + 49$$
$$= 52 \quad \text{which is divisible by 4.}$$

assume ~~show~~ true for $n = k$

i.e $3^k + 7^{k+1} = 4Q$ where Q is an integer.

$$3^k = 4Q - 7^{k+1}$$

prove true for $n = k+1$

$$3^{k+1} + 7^{(k+1)+1} = 3.3^k + 7.7^{k+1}$$
$$= 3(4Q - 7^{k+1}) + 7.7^{k+1}$$
$$= 4.3Q - 3.7^{k+1} + 7.7^{k+1}$$
$$= 4.3Q + 4.7^{k+1}$$
$$= 4(3Q + 7^{k+1})$$

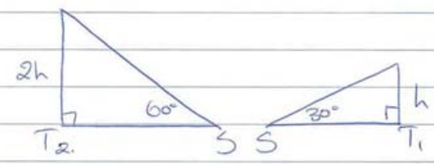
which is divisible by 4.

Hence, by induction $3^n + 7^{n+1}$ is divisible by 4 for $n \geq 1$.

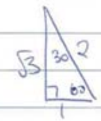
d. $\int_0^{\frac{\pi}{4}} \sin^2 2x \, dx$ $\quad 1 - 2\sin^2 x = \cos 2x$ 3.

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1 - \cos 4x)\, dx$$
$$= \frac{1}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{4}\sin\frac{4\pi}{4} - \left(0 - \frac{1}{4}\sin 4(0)\right)\right]$$
$$= \frac{1}{8}(\pi - 0) = \frac{\pi}{8}$$

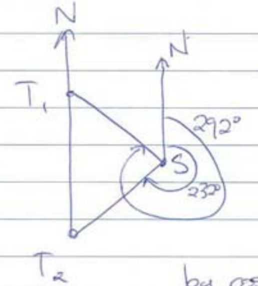
e. 4.



$$\tan 60° = \frac{2h}{ST_2} \qquad \tan 30° = \frac{h}{ST_1}$$



$$ST_2 = \frac{2h}{\sqrt{3}} \qquad ST_1 = \sqrt{3}h$$



$$\angle T_1ST_2 = 292 - 232$$
$$= 60°$$

by cosine rule.

$$(T_1T_2)^2 = \left(\frac{2h}{\sqrt{3}}\right)^2 + (\sqrt{3}h)^2 - 2\left(\frac{2h}{\sqrt{3}}\right)(\sqrt{3}h)\cos 60°$$
$$= \frac{4h^2}{3} + 3h^2 - 4h^2 \cdot \frac{1}{2}$$
$$= \frac{4h^2 + 9h^2 - 6h^2}{3}$$
$$= \frac{7h^2}{3}$$
$$T_1T_2 = \sqrt{\frac{7h^2}{3}}$$
$$= \sqrt{\frac{7}{3}}\, h \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{21}\,h}{3} \text{ metres}$$

Q14. a. $f(x) = \sin^2 x - x + 1$ $\quad x_1 = \frac{\pi}{2}$ 2.

$f'(x) = 2\sin x \cos x - 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= \frac{\pi}{2} - \frac{\sin^2\frac{\pi}{2} - \frac{\pi}{2} + 1}{2\sin\frac{\pi}{2}\cos\frac{\pi}{2} - 1}$$

$$= \frac{\pi}{2} - \frac{1^2 - \frac{\pi}{2} + 1}{2(1)(0) - 1}$$

$$= \frac{\pi}{2} + \left(2 - \frac{\pi}{2}\right)$$

$$= 2.$$

b. $u = \tan x$ 3.

$du = \sec^2 x \, dx$

$$\int \frac{\sec^2 x}{\tan^2 x + 3}\,dx = \int \frac{du}{u^2+3}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}} + c$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x}{\sqrt{3}}\right) + c$$

c. i. $c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n = (1+x)^n$ 2.

integrating:

$$\frac{c_0 x}{1} + \frac{c_1 x^2}{2} + \frac{c_2 x^3}{3} + \ldots + \frac{c_n x^{n+1}}{n+1} = \frac{(1+x)^{n+1}}{n+1}$$

integrating:

$$\frac{c_0 x^2}{1.2} + \frac{c_1 x^3}{2.3} + \frac{c_2 x^4}{3.4} + \ldots + \frac{c_n x^{n+2}}{(n+1)(n+2)} = \frac{(1+x)^{n+2}}{(n+1)(n+2)}$$

ii. let $x = -1$

$$\frac{c_0(-1)^2}{1.2} + \frac{c_1(-1)^3}{2.3} + \frac{c_2(-1)^4}{3.4} + \ldots + \frac{c_n(-1)^{n+2}}{(n+1)(n+2)} = \frac{(1-1)^{n+2}}{(n+1)(n+2)}$$

$$\frac{c_0}{1.2} - \frac{c_1}{2.3} + \frac{c_2}{3.4} + \ldots + \frac{(-1)^n(-1)^2 c_n}{(n+1)(n+2)} = 0$$

d. $x = vt\cos\theta$ — I

$y = vt\sin\theta - \frac{1}{2}gt^2$ — II

i. water is at ground level at $y=0$ 2.

$$vt\sin\theta - \frac{1}{2}gt^2 = 0$$

$$t\left(v\sin\theta - \frac{1}{2}gt\right) = 0$$

$t = 0$ or $\frac{1}{2}gt = v\sin\theta$

$$t = \frac{2v\sin\theta}{g}$$

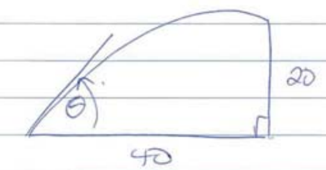
substitute into ~~IV~~ I

$$x = v\left(\frac{2v\sin\theta}{g}\right)\cos\theta$$

$$= \frac{v^2}{g} 2\sin\theta\cos\theta$$

$$= \frac{v^2 \sin 2\theta}{g}$$

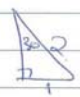
ii. 1.



when $\theta = 15°$, $x = 40$ m.

$$\therefore 40 = \frac{v^2 \sin 2(15)°}{g}$$

$$v^2 = \frac{40g}{\sin 30°}$$



$$= \frac{40g}{\frac{1}{2}}$$

$$\therefore v^2 = 80g.$$

iii. from I 2.

$$t = \frac{x}{v\cos\theta}$$

Subst into II

$$y = v\sin\theta\left(\frac{x}{v\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{1}{2}g\frac{x^2}{v^2\cos^2\theta}$$

$$= x\tan\theta - \frac{1}{2}g\frac{x^2\sec^2\theta}{80g}$$

$$\therefore y = x\tan\theta - \frac{x^2\sec^2\theta}{160}$$

iv. when $x = 40$, $y = 20$ water just clears the top. 2.

$$20 = 40\tan\theta - \frac{40^2\sec^2\theta}{160}$$

$$= 40\tan\theta - 10(1+\tan^2\theta)$$

$$2 = 4\tan\theta - 1 - \tan^2\theta.$$

$$\therefore \tan^2\theta - 4\tan\theta + 3 = 0$$